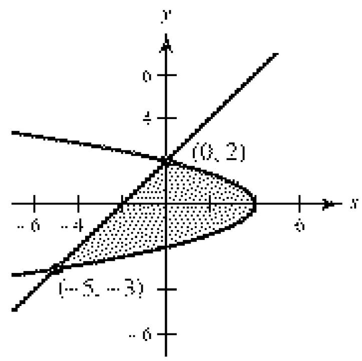
Summer Assignment for Calc III (M215)

Answer Section

1\.

a)



$$x = 4 - y^2$$
$$x = y - 2$$
$$4 - y^2 = y - 2$$
$$y^2 + y - 6 = 0$$
$$(y+3)(y-2) = 0$$

Intersection points $(0, 2)$ and $(-5, -3)$

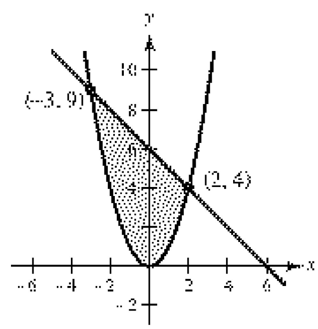
$$A = \int_{-5}^{0} \left[(x+2) + \sqrt{4-x} \right] dx + \int_{0}^{4} 2\sqrt{4-x}\ dx$$
$$= \frac{61}{6} + \frac{32}{3} = \frac{125}{6}$$

b) $A = \int_{-3}^{2} \left[\left(4-y^2\right) - \left(y-2\right) \right] dy \quad = \frac{125}{6}$

c) The second method is simpler. Explanations will vary.

2\.

a) $y = x^2$ and $y = 6 - x$



$x^2 = 6 - x \quad \Rightarrow \quad x^2 + x - 6 = 0$

$\Rightarrow \quad (x+3)(x-2) = 0$

Intersection points: $(2, 4)$ and $(-3, 9)$

$$A = \int_{-3}^{2} \left[(6-x) - x^2 \right] dx \quad = \frac{125}{6}$$

b) $A = \int_{0}^{4} 2\sqrt{y}\ dy + \int_{4}^{9} \left[\left(6-y\right) + \sqrt{y} \right] dy$

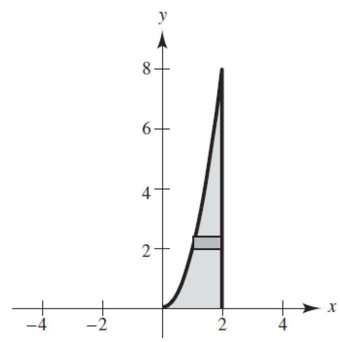
$$= \frac{32}{3} + \frac{61}{6} \quad = \frac{125}{6}$$

c) The first method is simpler. Explanations will vary.

3. $y = 2x^2, \quad y = 0, \quad x = 2$

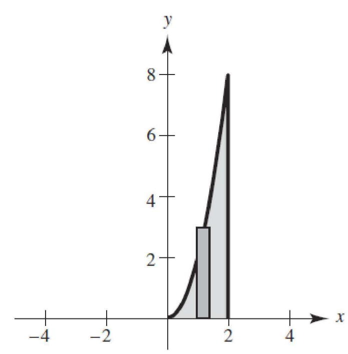
a) $R(y) = 2,\ r(y) = \sqrt{\dfrac{y}{2}}$

$$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right)dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$$



b) $R(x) = 2x^2,\ r(x) = 0$

$$V = \pi\int_0^2 4x^4\,dx = \pi\left[\frac{5x^5}{5}\right]_0^2 = \frac{128\pi}{5}$$



c) $R(x) = 8,\ r(x) = 8 - 2x^2$

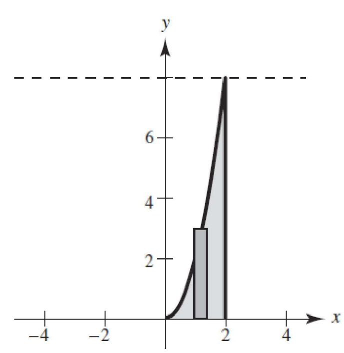
$$V = \pi\int_0^2 \left[64 - \left(64 - 32x^2 + 4x^4\right)\right]dx$$

$$= \pi\int_0^2 \left(32x^2 - 4x^4\right)dx$$

$$= 4\pi\int_0^2 \left(8x^2 - x^4\right)dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\pi}{15}$$

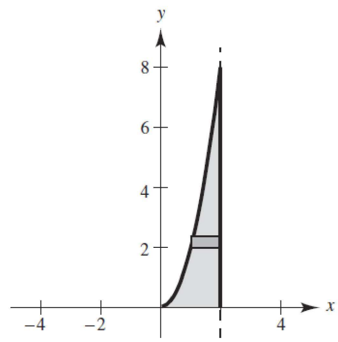


d) $R(y) = 2 - \sqrt{\frac{y}{2}}$, $r(y) = 0$

$$V = \pi \int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi \int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right)^2 dy$$

$$= \pi \left[4y - \frac{4\sqrt{2}}{3} y^{\frac{3}{2}} + \frac{y^2}{4}\right]_0^8 = \frac{16\pi}{3}$$



4. $\int_0^1 \sqrt{1 + 16x^6}\ dx$

5. $y = 2x^{\frac{3}{2}} + 3$

$y' = 3x^{\frac{1}{2}}$, $\quad 0 \le x \le 9$

$$s = \int_0^9 \sqrt{1 + 9x}\ dx$$

$$= \left[\frac{2}{27}(1 + 9x)^{\frac{3}{2}}\right]_0^9 = \frac{2}{27}\left(82^{\frac{3}{2}} - 1\right) \approx 54.929$$

6. $V = \pi \int_1^4 \left(\sqrt{x}\right)^2 dx = \pi \int_1^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_1^4 = \frac{15\pi}{2}$

7. $\int_0^2 y^3 dy$

8. $f(z) = 2z^2 - 3\sqrt{4 - z^2} + C$

9. $\sin x^3 + C$

10. 1

11. $\frac{x^5}{25}\left[5\ln(x) - 1\right] + C$

12. $\dfrac{\sin 3x\left(3-\sin^2 3x\right)}{9}+C$

$\int \cos^3(3x)dx\ldots$ let $u=3x$, so $du=3dx$

$$\int \cos^3(3x)dx=\int \frac{1}{3}\cos^3 u\,du=\frac{1}{3}\int \cos u\left(\cos^2 u\right)du=\frac{1}{3}\int \cos u\left(1-\sin^2 u\right)du=\frac{1}{3}\left(\int \cos u\,du-\int \sin^2 u\cos u\,du\right)$$

$$=\frac{1}{3}\left(\sin u-\frac{\sin^3 u}{3}\right)=\frac{1}{3}\left(\sin 3x-\frac{\sin^3 3x}{3}\right)=\frac{\sin 3x}{3}-\frac{\sin^3 3x}{9}=\frac{\sin 3x}{3}\left(1-\frac{\sin^2 3x}{3}\right)+C$$

13. $x=x_1+t\left(x_2-x_1\right)$

$y=y_1+t\left(y_2-y_1\right)$

$\therefore\quad t=\dfrac{x-x_1}{x_2-x_1}$

and $y=y_1+\left(\dfrac{x-x_1}{x_2-x_1}\right)\left(y_2-y_1\right)$

$y-y_1=\dfrac{y_2-y_1}{x_2-x_1}\left(x-x_1\right)$

$y-y_1=m\left(x-x_1\right)$

so $x=1+4t$ and $y=4-6t$

14. $x=h+r\cos\theta$

$y=k+r\sin\theta$

$\cos\theta=\dfrac{x-h}{r}$

$\sin\theta=\dfrac{y-k}{r}$

$\cos^2\theta+\sin^2\theta=\dfrac{(x-h)^2}{r^2}+\dfrac{\left(y-k\right)^2}{r^2}=1$

$(x-h)^2+\left(y-k\right)^2=r^2$

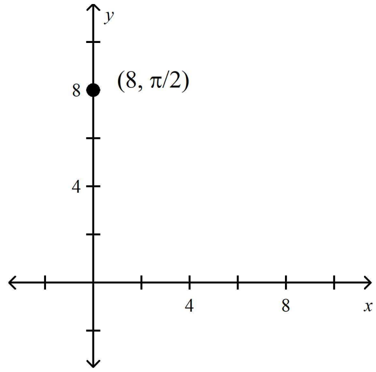
so $x=-6+4\cos\theta$ and $y=2+4\sin\theta$

15.

$$x = 8\cos\frac{\pi}{2} = 0$$

$$y = 8\sin\frac{\pi}{2} = 8$$

$$(x,y) = (0,8)$$

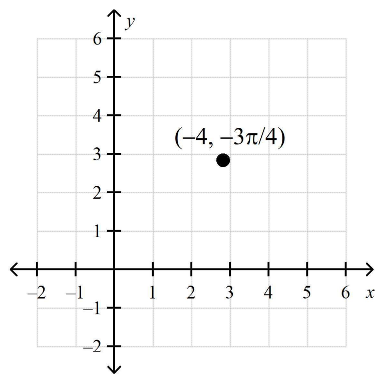


16.

$$x = -4\cos\left(\frac{-3\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$y = -4\sin\left(\frac{-3\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$(x,y) = \left(2\sqrt{2},\ 2\sqrt{2}\right)$$



17.

$$r = 3\sin\theta$$

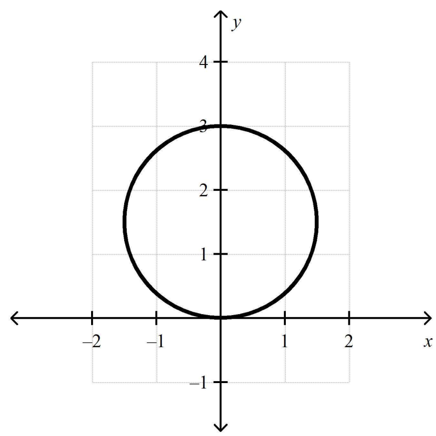
$$r^2 = 3r\sin\theta$$

$$x^2 + y^2 = 3y$$

$$x^2 + \left(y^2 - 3y + \frac{9}{4}\right) = \frac{9}{4}$$

$$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

a circle

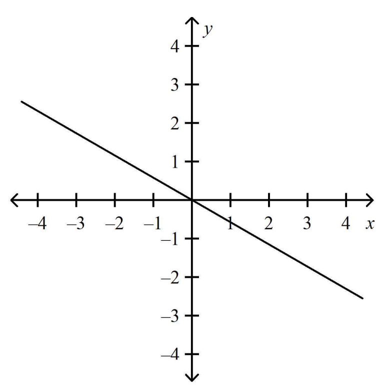


18.

$$\tan\theta = \tan\frac{5\pi}{6}$$

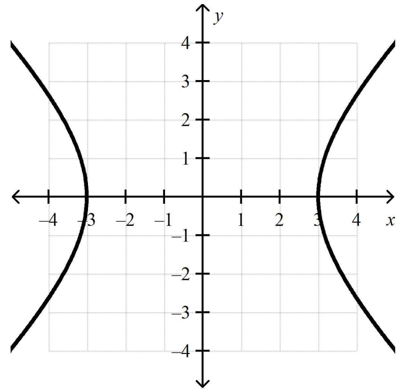
$$\frac{y}{x} = -\frac{\sqrt{3}}{3}$$

$$y = -\frac{\sqrt{3}}{3}x$$



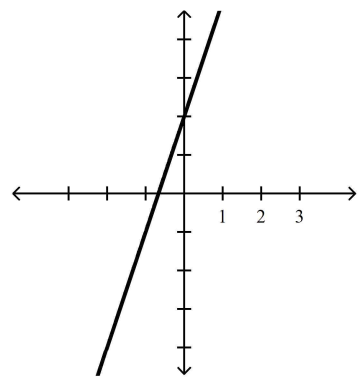
19.

$$x^2 - y^2 = 9$$
$$(r\cos\theta)^2 - (r\sin\theta)^2 = 9$$
$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 9$$
$$r^2\cos 2\theta = 9$$
$$r = \frac{3}{\sqrt{\cos 2\theta}}$$



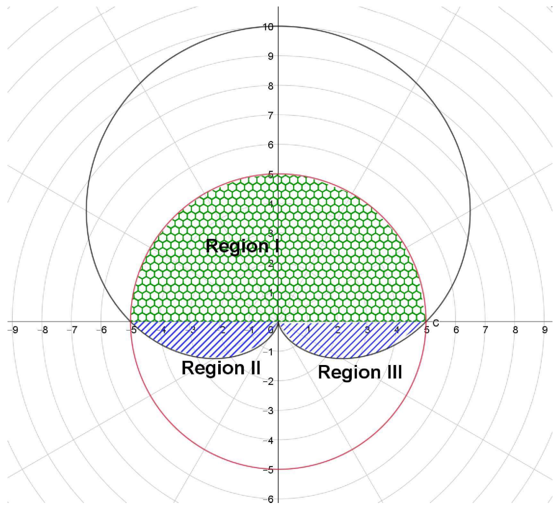
20.

$$3x - y + 2 = 0$$
$$3r\cos\theta - r\sin\theta + 2 = 0$$
$$r(3\cos\theta - \sin\theta) = -2$$
$$r = \frac{-2}{3\cos\theta - \sin\theta}$$



21. $r = 5$ and $r = 5(1+\sin\theta)$ intersect at $\sin\theta = 0 \quad \rightarrow \quad \theta = 0$ or $\theta = \pi$.

Dividing the region into three subregions we have:



Subregion 1 is a semicircle, so the area is: $A = \dfrac{\pi r^2}{2} = \dfrac{\pi\, 25}{2}$

For subregions 2 and 3 the area is: $A = \displaystyle\int_{\pi}^{2\pi} \frac{1}{2}\left[f(\theta)\right]^2 d\theta$

$$= \int_{\pi}^{2\pi} \frac{1}{2}\left[5(1+\sin\theta)\right]^2 d\theta$$

$$= \frac{25}{2}\int_{\pi}^{2\pi} (1+\sin\theta)^2\, d\theta$$

$$= \frac{25}{2}\int_{\pi}^{2\pi} \left(1+2\sin\theta+\sin^2\theta\right) d\theta$$

$$= \frac{25}{2}\left[\int_{\pi}^{2\pi} d\theta + 2\int_{\pi}^{2\pi} \sin\theta\, d\theta + \int_{\pi}^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{25}{2}\left[\theta + 2(-\cos\theta) + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi}^{2\pi}$$

$$= \frac{25}{2}\left[\left(2\pi - 2(1) + \frac{2\pi}{2} - \frac{1}{4}(0)\right) - \left(\pi - 2(-1) + \frac{\pi}{2} - \frac{1}{4}(0)\right)\right]$$

$$= \frac{25}{2}\left[2\pi - 2 + \pi - \pi - 2 - \frac{\pi}{2}\right]$$

$$= \frac{25}{2}\left(\frac{3\pi}{2} - 4\right) = \frac{25}{2}\left(\frac{3\pi-8}{2}\right) = \frac{25}{4}(3\pi - 8)$$

So the total area is $\text{Total Area} = \dfrac{25}{2}\pi + \dfrac{25}{4}(3\pi-8) = \dfrac{25}{4}(2\pi + 3\pi - 8) = \dfrac{25}{4}(5\pi - 8)$

22. $y = \dfrac{2}{x-7} - 9$

23. $\dfrac{dy}{dx} = -12t^{\frac{11}{12}}$

24. $\mathbf{v} = 4\mathbf{i} - 5\mathbf{j}$

25. $\left\langle 27, -36 \right\rangle$