Summer Assignment for Calc III (M215)

Answer Section

PROBLEM

1. $A = \int_{-2}^{2} \left[(2x+5) - \left(x^2 + 2x + 1 \right) \right] dx$

   $= \int_{-2}^{2} \left(-x^2 + 4 \right) dx$

2. $A = 2\int_{-1}^{0} 3\left(x^3 - x \right) d\,x$

   $= 6\int_{-1}^{0} \left(x^3 - x \right) dx$

   or $A = -6\int_{0}^{1} \left(x^3 - x \right) dx$

3. $x = x_1 + t\left(x_2 - x_1 \right)$

   $y = y_1 + t\left(y_2 - y_1 \right)$

   $\therefore \quad t = \dfrac{x - x_1}{x_2 - x_1}$

   and $y = y_1 + \left(\dfrac{x - x_1}{x_2 - x_1} \right)\left(y_2 - y_1 \right)$

   $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1} \left(x - x_1 \right)$

   $y - y_1 = m\left(x - x_1 \right)$

   so $x = 1 + 4t$ and $y = 4 - 6t$

4. $x = h + r\cos\theta$

$y = k + r\sin\theta$

$\cos\theta = \dfrac{x-h}{r}$

$\sin\theta = \dfrac{y-k}{r}$

$\cos^2\theta + \sin^2\theta = \dfrac{(x-h)^2}{r^2} + \dfrac{(y-k)^2}{r^2} = 1$

$(x-h)^2 + (y-k)^2 = r^2$

so $x = -6 + 4\cos\theta$ and $y = 2 + 4\sin\theta$

5.

a)

$x = 4 - y^2$

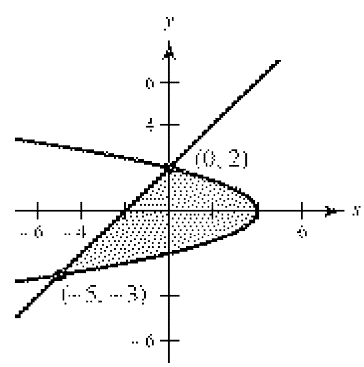
$x = y - 2$

$4 - y^2 = y - 2$

$y^2 + y - 6 = 0$

$(y+3)(y-2) = 0$

Intersection points $(0, 2)$ and $(-5, -3)$



$A = \int_{-5}^{0} \left[(x+2) + \sqrt{4-x} \right] dx + \int_{0}^{4} 2\sqrt{4-x}\ dx$

$= \dfrac{61}{6} + \dfrac{32}{3} = \dfrac{125}{6}$

b) $A = \int_{-3}^{2} \left[(4-y^2) - (y-2) \right] dy = \dfrac{125}{6}$

c) The second method is simpler. Explanations will vary.

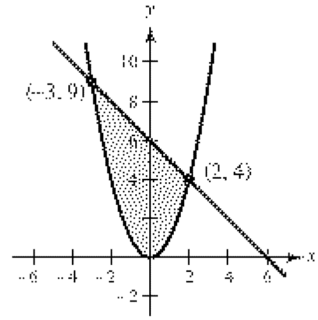
6.

a) $y = x^2$ and $y = 6 - x$

$x^2 = 6 - x \quad \Rightarrow \quad x^2 + x - 6 = 0$

$\Rightarrow \quad (x+3)(x-2) = 0$

Intersection points: $(2, 4)$ and $(-3, 9)$



$$A = \int_{-3}^{2} \left[(6-x) - x^2 \right] dx \;\; = \frac{125}{6}$$

b) $$A = \int_{0}^{4} 2\sqrt{y}\, dy + \int_{4}^{9} \left[\left(6-y\right) + \sqrt{y} \right] dy$$

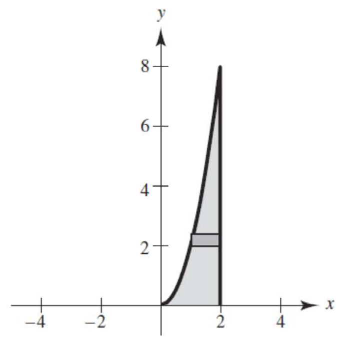
$$= \frac{32}{3} + \frac{61}{6} \;\; = \frac{125}{6}$$

c) The first method is simpler. Explanations will vary.

7. $y = 2x^2, \quad y = 0, \quad x = 2$

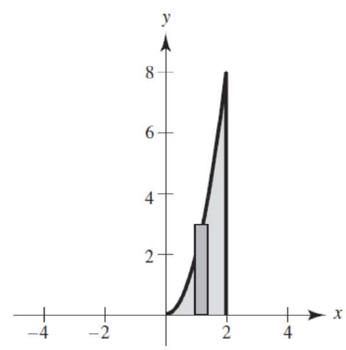
a) $R(y) = 2,\ r(y) = \sqrt{\dfrac{y}{2}}$

$$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right)dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$$



b) $R(x) = 2x^2,\ r(x) = 0$

$$V = \pi\int_0^2 4x^4\,dx = \pi\left[\frac{5x^5}{5}\right]_0^2 = \frac{128\pi}{5}$$



c) $R(x) = 8,\ r(x) = 8 - 2x^2$

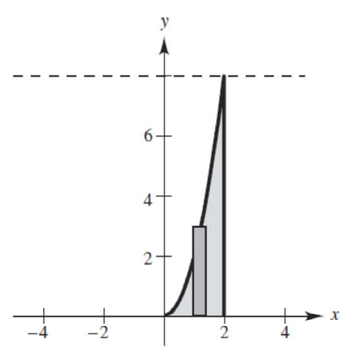
$$V = \pi\int_0^2 \left[64 - \left(64 - 32x^2 + 4x^4\right)\right]dx$$

$$= \pi\int_0^2 \left(32x^2 - 4x^4\right)dx$$

$$= 4\pi\int_0^2 \left(8x^2 - x^4\right)dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\,\pi}{15}$$

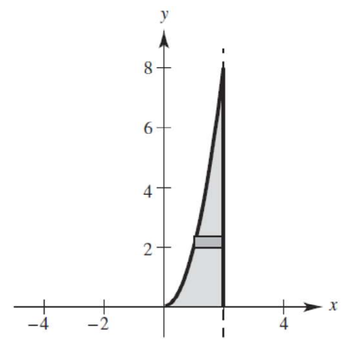


d) $R(y) = 2 - \sqrt{\dfrac{y}{2}},\ r(y) = 0$

$$V = \pi \int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi \int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right)^2 dy$$

$$= \pi \left[4y - \frac{4\sqrt{2}}{3} y^{\frac{3}{2}} + \frac{y^2}{4}\right]_0^8 = \frac{16\pi}{3}$$

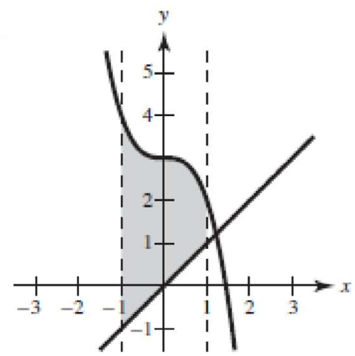


8.

$$A = \int_{-1}^{1} \left[\left(-x^3 + 3\right) - x\right] dx$$

$$= \left[\frac{-x^4}{4} + 3x - \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left(-\frac{1}{4} + 3 - \frac{1}{2}\right) - \left(-\frac{1}{4} - 3 - \frac{1}{2}\right) = 6$$



9. The points of intersection are given by:

$$-x^2 + 4x + 1 = x + 1$$

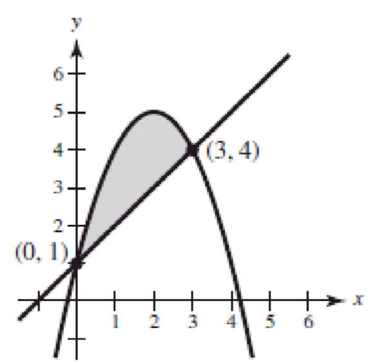
$$-x^2 + 3x = 0$$

$$x^2 = 3x \quad \text{when } x = 0,\ 3$$

$$A = \int_0^3 \left[\left(-x^2 + 4x + 1\right) - (x + 1)\right] dx$$

$$= \int_0^3 \left(-x^2 + 3x\right) dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = -9 + \frac{27}{2} = \frac{9}{2}$$



10. $\displaystyle\int_0^1 \sqrt{1 + 16x^6}\ dx$

11. $y = 2x^{\frac{3}{2}} + 3$

$y' = 3x^{\frac{1}{2}}, \qquad 0 \le x \le 9$

$s = \int_0^9 \sqrt{1+9x}\ dx$

$= \left[\frac{2}{27}(1+9x)^{\frac{3}{2}} \right]_0^9 = \frac{2}{27}\left(82^{\frac{3}{2}} - 1 \right) \approx 54.929$

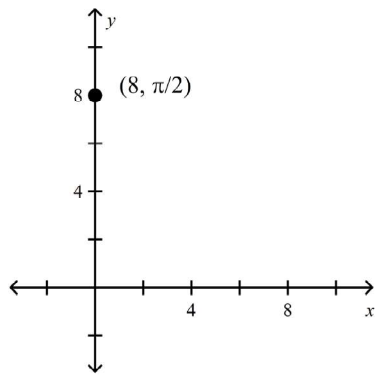
12. $V = \pi \int_1^4 \left(\sqrt{x} \right)^2 dx = \pi \int_1^4 x\,dx = \pi \left[\frac{x^2}{2} \right]_1^4 = \frac{15\pi}{2}$

13.

$x = 8\cos\frac{\pi}{2} = 0$

$y = 8\sin\frac{\pi}{2} = 8$

$(x, y) = (0, 8)$

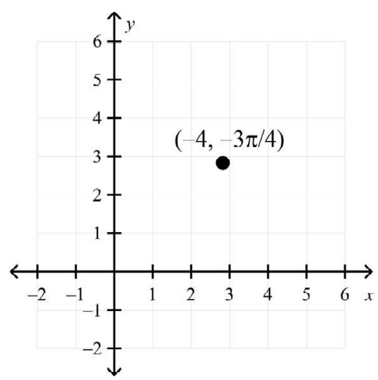


14.

$x = -4\cos\left(\frac{-3\pi}{4} \right) = -4\left(-\frac{\sqrt{2}}{2} \right) = 2\sqrt{2}$

$y = -4\sin\left(\frac{-3\pi}{4} \right) = -4\left(-\frac{\sqrt{2}}{2} \right) = 2\sqrt{2}$

$(x, y) = \left(2\sqrt{2},\ 2\sqrt{2} \right)$



15.

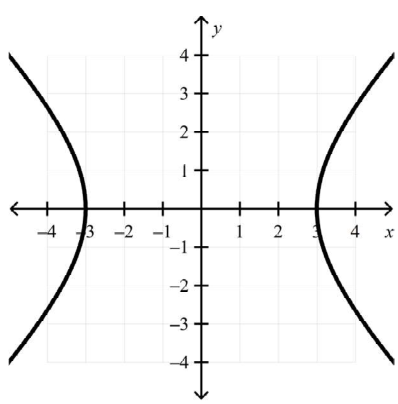
$x^2 - y^2 = 9$

$(r\cos\theta)^2 - (r\sin\theta)^2 = 9$

$r^2\left(\cos^2\theta - \sin^2\theta \right) = 9$

$r^2 \cos 2\theta = 9$

$r = \frac{3}{\sqrt{\cos 2\theta}}$



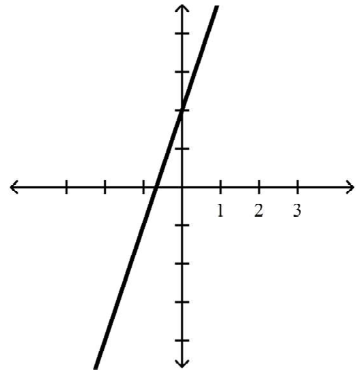
16.

$$3x - y + 2 = 0$$

$$3r\cos\theta - r\sin\theta + 2 = 0$$

$$r(3\cos\theta - \sin\theta) = -2$$

$$r = \frac{-2}{3\cos\theta - \sin\theta}$$



17.

$$r = 3\sin\theta$$

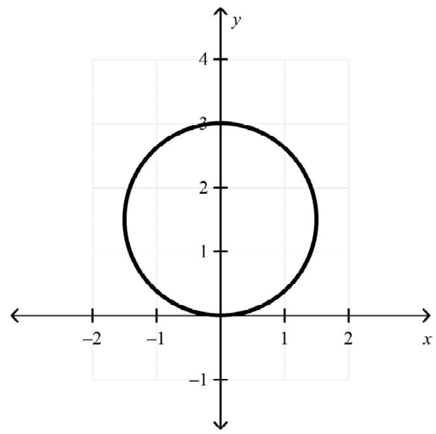
$$r^2 = 3r\sin\theta$$

$$x^2 + y^2 = 3y$$

$$x^2 + \left(y^2 - 3y + \frac{9}{4}\right) = \frac{9}{4}$$

$$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

a circle

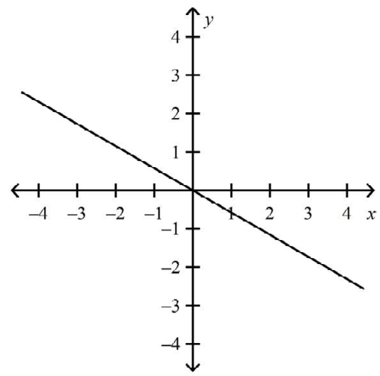


18.

$$\tan\theta = \tan\frac{5\pi}{6}$$

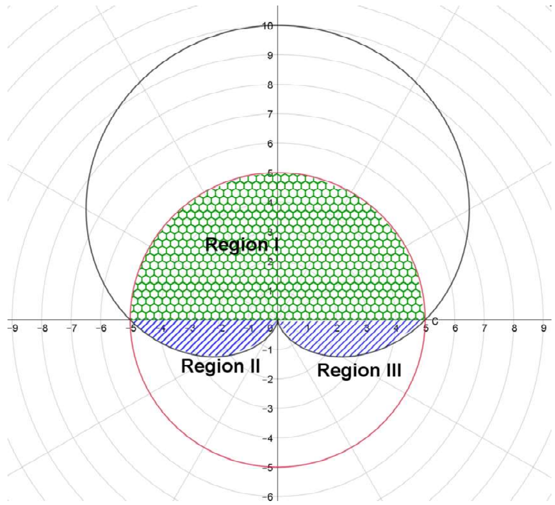
$$\frac{y}{x} = -\frac{\sqrt{3}}{3}$$

$$y = -\frac{\sqrt{3}}{3}x$$



19. $r = 5$ and $r = 5(1+\sin\theta)$ intersect at $\sin\theta = 0 \quad \rightarrow \quad \theta = 0$ or $\theta = \pi$.

Dividing the region into three subregions we have:



Subregion 1 is a semicircle, so the area is: $A = \dfrac{\pi r^2}{2} = \dfrac{\pi\, 25}{2}$

For subregions 2 and 3 the area is: $A = \displaystyle\int_{\pi}^{2\pi} \frac{1}{2}\left[f(\theta)\right]^2 d\theta$

$$= \int_{\pi}^{2\pi} \frac{1}{2}\left[5(1+\sin\theta)\right]^2 d\theta$$

$$= \frac{25}{2}\int_{\pi}^{2\pi} (1+\sin\theta)^2\, d\theta$$

$$= \frac{25}{2}\int_{\pi}^{2\pi} \left(1+2\sin\theta+\sin^2\theta\right) d\theta$$

$$= \frac{25}{2}\left[\int_{\pi}^{2\pi} d\theta + 2\int_{\pi}^{2\pi} \sin\theta\, d\theta + \int_{\pi}^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{25}{2}\left[\theta + 2(-\cos\theta) + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi}^{2\pi}$$

$$= \frac{25}{2}\left[\left(2\pi - 2(1) + \frac{2\pi}{2} - \frac{1}{4}(0)\right) - \left(\pi - 2(-1) + \frac{\pi}{2} - \frac{1}{4}(0)\right)\right]$$

$$= \frac{25}{2}\left[2\pi - 2 + \pi - \pi - 2 - \frac{\pi}{2}\right]$$

$$= \frac{25}{2}\left(\frac{3\pi}{2} - 4\right) = \frac{25}{2}\left(\frac{3\pi-8}{2}\right) = \frac{25}{4}(3\pi - 8)$$

So the total area is $\text{Total Area} = \dfrac{25}{2}\pi + \dfrac{25}{4}(3\pi-8) = \dfrac{25}{4}(2\pi+3\pi-8) = \dfrac{25}{4}(5\pi-8)$